HT10 Analysis II Sheet 1

This sheet is about function limits ($\epsilon - \delta$ definition and relation to sequence limits) and the definition of continuity.

1. Suppose that f, g are real-valued functions defined on some interval (a, b) containing the point x_0, and that $\lim_{x\to x_0} f(x) = k$ and $\lim_{x\to x_0} g(x) = l$.

(a) Using the $\epsilon - \delta$ definition of $\phi(x) \to \alpha$, prove that $\lim_{x\to x_0} (f(x) + g(x)) = k + l$.

(b) Using the the theorem relating function limits to sequence limits prove that $\lim_{x\to x_0} f(x)g(x) = kl$.

(c) Prove that if $f(x) < g(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ (for some $\delta > 0$) then $k \leqslant l$. In this case is it always true that $k < l$?

(d)[*optional/revision*] Prove the other 'Algebra of Limits' results in §1.7.

2. Suppose that (i) $f(x) \to y_0$ as $x \to x_0$, (ii) $g(y) \to l$ as $y \to y_0$, and (iii) $g(y_0) = l$. Prove carefully that $g(f(x)) \to l$ as $x \to x_0$.

3. (a) Prove that if f is continuous at x_0, then $|f|$ is continuous at x_0.

(b) For x, $y \in \mathbb{R}$, show that $\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$.

(c) Prove that if f and g are continuous at x_0, then $\max\{f, g\}$ is continuous at x_0.

(d)[*optional*] What about $\min\{f, g\}$?

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x + y) = f(x) + f(y) \qquad \text{for any } x, y \in \mathbb{R}.$$

Using elementary algebra prove that $f(0) = 0$ and $f(-x) = -f(x)$ for every x; by induction prove that $f(nx) = nf(x)$ for every x and every natural number n; and then show that $f(rx) = rf(x)$ for every x and any rational number $r = n/m$. Prove that if f is also continuous then, for some constant c,

$$f(x) = cx \qquad \text{for every } x \in \mathbb{R}.$$

[*This is now a natural question: are there any* non-continuous *functions which satisfy the hypotheses? There are, but to prove this is still beyond our reach.*]

5. (a) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at the points of the set $\{1/n : n \text{ a positive integer}\} \cup \{0\}$ but is continuous everywhere else.

(b) Find a function $g : \mathbb{R} \to \mathbb{R}$ which is discontinuous at the points of the set $\{1/n : n \text{ a positive integer}\}$ but is continuous everywhere else.

(c) Let the function $f : (0, 1] \to \mathbb{R}$ be given by $f(x) = 0$ if x is irrational, and $f(x) = \frac{1}{p+q}$ if $x = p/q \in (0, 1]$ in lowest terms. Prove that it is continuous at $\frac{1}{\sqrt{2}}$.

J. D. HT 10